

## Article

# Lie Point Symmetries, Traveling Wave Solutions and Conservation Laws of a Non-linear Viscoelastic Wave Equation

Almudena P. Márquez \*  and María S. Bruzón 

Department of Mathematics, University of Cadiz, Puerto Real, 11510 Cadiz, Spain; m.bruzon@uca.es

\* Correspondence: almudena.marquez@uca.es

**Abstract:** This paper studies a non-linear viscoelastic wave equation, with non-linear damping and source terms, from the point of view of the Lie groups theory. Firstly, we apply Lie's symmetries method to the partial differential equation to classify the Lie point symmetries. Afterwards, we reduce the partial differential equation to some ordinary differential equations, by using the symmetries. Therefore, new analytical solutions are found from the ordinary differential equations. Finally, we derive low-order conservation laws, depending on the form of the damping and source terms, and discuss their physical meaning.

**Keywords:** viscoelastic wave equation; Lie symmetries; traveling wave solutions; conservation laws



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## 1. Introduction

Lately, many viscoelastic wave equations have been considered in the literature. The single viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^1 h(t-s) \Delta u(x,s) ds + f(u_t) = g(u)$$

in  $\Omega \times (0, \infty)$ , where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  ( $N \geq 1$ ) with initial and boundary conditions, has been extensively studied. Several results concerning non-existence and blow-up solutions in finite time have been proved [1–8].

Furthermore, the non-linear viscoelastic wave equation with damping and source terms

$$u_{tt} - \Delta u + f(u_t) = g(u), \quad x \in \Omega, t > 0, \quad (1)$$

has also been very studied obtaining similar results [9,10]. As in the single viscoelastic wave equation, it is well-known that the damping term  $f(u_t)$  assures global existence in the absence of the source term ( $g(u) = 0$ ). The interaction between the damping term and the source term makes the problem more interesting.

Moreover, Messaoudi [11] considered the non-linear viscoelastic wave equation with damping and source terms

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad x \in \Omega, t > 0.$$

For this equation, Georgiev and Todorova [12] and Messaoudi [13] analyzed blow-up solutions in different situations.

In general, many authors showed interest in these viscoelastic wave equations. However, in this paper, we focus on studying the viscoelastic wave Equation (1) but from a point of view of the Lie groups theory. In fact, we have published a previous work analyzing this model [14]. Moreover, in this paper we present new results for the model. It is found a complete classification of Lie point symmetries with its associated reductions, new soliton-type solutions, and a complete classification of multipliers and conservation laws with a discussion of their physical meaning.

The resolution of non-linear partial differential equations (PDEs) is a very important field of research in applied mathematics. Symmetry reductions and analytical solutions have many applications in the context of differential equations. For instance, analytical solutions arising from symmetry methods can be used to study properties, such as asymptotic and blow-up behavior. A large amount of literature has been published about the application of the Lie transformation group theory to construct solutions of non-linear PDEs [15–20]. For example, the Fisher equation was studied in [21] to find new analytical solutions. In [22], a  $(2 + 1)$ -dimensional Zakharov-Kuznetsov equation was also investigated using Lie symmetry analysis. The authors of [23] analyzed a system of dispersive evolution equations to obtain new exact solutions too. The symmetries leaving the equation invariant can reduce the number of independent variables, transforming the PDEs into ordinary differential equations (ODEs), which are generally easier to solve.

Additionally, there is a similar one-dimensional equation called the “good” Boussinesq equation considering  $\Delta^2$ . This change would transform the original second-order PDE (1) to a 4-th order PDE, complicating it but of interest. Nevertheless, there have been a few numerical works of this equation in recent years, such as applying a Fourier pseudo-spectral method [24], and a 2-nd order operator splitting numerical scheme for the Equation [25]. The stability and convergence estimates have been presented in these works.

Conservation laws analyze which physical properties of a PDE do not change in the course of time. In particular, local conservation laws are continuity equations yielding conserved quantities of physical importance for all solutions of a given equation. For any PDE, a complete classification of conservation laws can be determined by the multiplier method [26,27]. In [28], the authors obtained the conservation laws and discussed the physical meaning of the corresponding conserved quantities. A classification of conservation laws of a generalized quasilinear KdV equation was provided in [29] too. Moreover, a  $(1 + 1)$ -dimensional coupled modified KdV-type system was studied in [30], constructing its conservation laws also using the multiplier method.

To sum up, the aim of this work is to do a complete Lie group classification of Equation (1). Afterwards, we present the reductions obtained from the different symmetries, transforming the PDE into ODEs. Moreover, we obtain traveling wave solutions by comparing Equation (1) and similar equations studied previously [31–33]. Finally, we give a complete classification of the conservation laws admitted by Equation (1).

The structure of the paper is as follows: In Section 2, we study the Lie point symmetries of Equation (1), and in Section 3, we obtain the symmetry reductions, the symmetry variables, and the reduced ODEs. Next, in Section 4, we construct traveling wave solutions using the reduced equations. Then, in Section 5, we present a classification of the conservation laws and the multipliers of Equation (1). Finally, in Section 6, we give some conclusions of the work.

## 2. Lie Point Symmetries

The idea of the Lie groups theory of symmetry analysis of differential equations relies on the invariance of the equation under a transformation of independent and dependent variables. This transformation sets up a local group of point transformations yielding to a diffeomorphism on the space of independent and dependent variables, so the solutions of the original equation map to other solutions. Any transformation of the independent and dependent variables leads to a transformation of the derivatives [34].

The application of the Lie groups theory to differential equations is completely algorithmic. However, it usually involves many tedious calculations. Nevertheless, we make use of powerful softwares, such as Maple and the needed calculations are done rapidly. Applying the classical Lie method to search for symmetries provides a set of different expressions for the unknown functions  $f(u_t)$  and  $g(u)$ , for which the equation admits symmetries.

In this section, let us briefly describe the classical Lie method and its application to Equation (1), obtaining the symmetry reductions, the symmetry variables and the reduced

equations. For details, this method is described in excellent textbooks, such as [17,19] and references therein.

It is consider the one-parameter Lie group of infinitesimal transformations in  $(x, t, u)$  given by

$$\begin{aligned}x^* &= x + \epsilon \zeta(x, t, u) + \mathcal{O}(\epsilon^2), \\t^* &= t + \epsilon \tau(x, t, u) + \mathcal{O}(\epsilon^2), \\u^* &= u + \epsilon \eta(x, t, u) + \mathcal{O}(\epsilon^2),\end{aligned}\quad (2)$$

where  $\epsilon$  is the group parameter.

The infinitesimal point symmetries constitute the infinitesimal generator

$$\mathbf{v} = \zeta(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \quad (3)$$

Each infinitesimal generator (Equation (3)) is associated with a transformation, determined by solving the system of ODEs

$$\frac{\partial \hat{x}}{\partial \epsilon} = \zeta(\hat{x}, \hat{t}, \hat{u}), \quad \frac{\partial \hat{t}}{\partial \epsilon} = \tau(\hat{x}, \hat{t}, \hat{u}), \quad \frac{\partial \hat{u}}{\partial \epsilon} = \eta(\hat{x}, \hat{t}, \hat{u}),$$

satisfying the initial conditions

$$\hat{x}|_{\epsilon=0} = x, \quad \hat{t}|_{\epsilon=0} = t, \quad \hat{u}|_{\epsilon=0} = u,$$

where  $\epsilon$  is the group parameter.

We point out that (3) is a point symmetry of Equation (1) if the 2-nd order prolongation of (3) leaves invariant Equation (1). This leads to an overdetermined linear system of determining equations for the infinitesimals  $\zeta(x, t, u)$ ,  $\tau(x, t, u)$  and  $\eta(x, t, u)$ , generated by applying the symmetry invariance condition

$$\text{pr}^{(2)}\mathbf{v}(u_{tt} - u_{xx} + f(u_t) - g(u)) = 0, \quad \text{when} \quad u_{tt} - u_{xx} + f(u_t) - g(u) = 0, \quad (4)$$

Here  $\text{pr}^{(2)}\mathbf{v}$  represents the second order prolongation of the vector field  $\mathbf{v}$ , defined by

$$\text{pr}^{(2)}\mathbf{v} = \mathbf{v} + \eta_x \frac{\partial}{\partial u_x} + \eta_t \frac{\partial}{\partial u_t} + \eta_{xx} \frac{\partial}{\partial u_{xx}} + \eta_{xt} \frac{\partial}{\partial u_{xt}} + \eta_{tt} \frac{\partial}{\partial u_{tt}},$$

where the coefficients are given by

$$\begin{aligned}\eta_x &= D_x \eta - u_t D_x \tau - u_x D_x \zeta, \\ \eta_t &= D_t \eta - u_t D_t \tau - u_x D_t \zeta, \\ \eta_{xx} &= D_x(\eta_x) - u_{xt} D_x \tau - u_{xx} D_x \zeta, \\ \eta_{xt} &= D_t(\eta_x) - u_{xt} D_x \tau - u_{xx} D_t \zeta, \\ \eta_{tt} &= D_t(\eta_t) - u_{tt} D_t \tau - u_{xt} D_t \zeta,\end{aligned}$$

with  $D_x$  and  $D_t$  the total derivatives of  $x$  and  $t$ , respectively.

The symmetry determining Equation (4) splits with respect to the  $t$ -derivatives and  $x$ -derivatives of  $u$ , getting an over-determined linear system of equations for the infinitesimals. Here Maple is used for defining the determining equations and then, the commands “rifsimp”, “dsolve” and “pdsolve” are used to solve the system. The command “rifsimp” gives a tree containing all solution cases. For each solution case, we use the commands “dsolve” and “pdsolve” to obtain solutions for the over-determined system. Therefore, we proceed to show the classification of all solution cases in Theorem 1.

**Theorem 1.** The Lie point symmetries admitted by the non-linear viscoelastic wave Equation (1) for  $f(u_t)$  and  $g(u)$  arbitrary functions, are generated by the transformations

$$\begin{aligned} \mathbf{v}_1 &= \partial_x, \\ (\hat{x}, \hat{t}, \hat{u})_1 &= (x + \epsilon, t, u), \quad \text{space translation,} \\ \mathbf{v}_2 &= \partial_t, \\ (\hat{x}, \hat{t}, \hat{u})_2 &= (x, t + \epsilon, u), \quad \text{time translation.} \end{aligned}$$

For some particular functions of  $f(u_t)$  and  $g(u)$ , the non-linear viscoelastic wave Equation (1) admits additional Lie point symmetries, given below.

1. For  $f(u_t) = -e^{-n}u_t^n + f_1$  and  $g(u) = ((g_0 u + g_1)(\frac{2}{n} - 1))^{\frac{n}{2-n}} - f_1$ , with  $n \neq 0, 1, 2$  and  $g_0, g_1, f_1$  arbitrary constants,

$$\begin{aligned} \mathbf{v}_3^1 &= g_0 \left( \frac{n-1}{n-2} \right) x \partial_x + g_0 \left( \frac{n-1}{n-2} \right) t \partial_t + (g_0 u + g_1) \partial_u, \\ (\hat{x}, \hat{t}, \hat{u})_3^1 &= (x e^{\frac{g_0(n-1)\epsilon}{n-2}}, t e^{\frac{g_0(n-1)\epsilon}{n-2}}, e^{g_0\epsilon} u + e^{g_0\epsilon} \int_0^\epsilon g_1 e^{-g_0 z_1} dz_1), \quad \text{scaling and shift.} \end{aligned}$$

2. For  $f(u_t) = au_t^2$  and  $g(u) = ke^{cu}$ , with  $c \neq 0$  and  $a, k$  arbitrary constants,

$$\begin{aligned} \mathbf{v}_3^2 &= -\frac{1}{2}cx\partial_x - \frac{1}{2}ct\partial_t + \partial_u, \\ (\hat{x}, \hat{t}, \hat{u})_3^2 &= (xe^{-1/2c\epsilon}, te^{-1/2c\epsilon}, \epsilon + u), \quad \text{scaling in } t \text{ and } x \text{ combined.} \end{aligned}$$

3. For  $f(u_t) = k$  and  $g(u) = (au + b)^n - k$ , with  $n \neq 1$ ,  $c \neq 0$  and  $a, b, k$  arbitrary constants,

$$\begin{aligned} \mathbf{v}_3^3 &= \frac{1}{2}a(n-1)x\partial_x + \frac{1}{2}a(n-1)t\partial_t + (au + b)\partial_u, \\ (\hat{x}, \hat{t}, \hat{u})_3^3 &= (e^{1/2ak_3(n-1)\epsilon} x + \int_0^\epsilon k_1 e^{-1/2ak_3(n-1)z_1} dz_1 e^{1/2ak_3(n-1)\epsilon}, \\ &\quad e^{1/2ak_3(n-1)\epsilon} t + \int_0^\epsilon k_2 e^{-1/2ak_3(n-1)z_1} dz_1 e^{1/2ak_3(n-1)\epsilon}, \\ &\quad e^{-ak_3\epsilon} u + \int_0^\epsilon -bk_3 e^{ak_3 z_1} dz_1 e^{-ak_3\epsilon}), \quad \text{scaling and shift.} \end{aligned}$$

4. For  $f(u_t) = au_t^{\frac{4}{3}}$  and  $g(u) = (cu + k)^2$ , with  $c \neq 0$  and  $a, k$  arbitrary constants,

$$\begin{aligned} \mathbf{v}_3^4 &= -\frac{1}{2}cx\partial_x - \frac{1}{2}ct\partial_t + (cu + k)\partial_u, \\ (\hat{x}, \hat{t}, \hat{u})_3^4 &= (xe^{-1/2c\epsilon}, te^{-1/2c\epsilon}, e^{c\epsilon} u + \int_0^\epsilon ke^{-cz_1} dz_1 e^{c\epsilon}), \quad \text{scaling and shift.} \end{aligned}$$

### 3. Symmetry Reductions

The symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi(x, t, u)u_x + \tau(x, t, u)u_t - \eta(x, t, u) = 0.$$

For Equation (1), a PDE with two independent variables, a single group reduction transforms the PDE into different ODEs.

**Reduction 1.** For the generator  $\lambda\mathbf{v}_1 + \mathbf{v}_2$ , we obtain the traveling wave reduction

$$z = x - \lambda t, \quad u(x, t) = h(z), \quad (5)$$

where  $h(z)$  satisfies

$$(\lambda^2 - 1)h'' + f(-\lambda h') - g(h) = 0. \quad (6)$$

**Reduction 2.** For the generator  $\mathbf{v}_3^1$ , we obtain the symmetry reduction

$$z = \frac{x}{t}, \quad u = x^{\frac{n-2}{n-1}} h(z) - \frac{g_1}{g_0},$$

where  $h(z)$  satisfies

$$\begin{aligned} & z^{\frac{3n^2+2n}{n^2-3n+2}} \left(-z^{\frac{1}{n-1}}\right)^{3n} \left( \left(g_0^{\frac{n}{n-2}} (2-n)^{\frac{n}{n-2}} (n-1)^2 e^{f_2 n} z^4 \right. \right. \\ & - g_0^{\frac{n}{n-2}} (2-n)^{\frac{n}{n-2}} (n-1)^2 e^{f_2 n} z^2 \left. \right) h^{\frac{n}{n-2} + \frac{2}{n-2}} h'' \\ & + \left( 2g_0^{\frac{n}{n-2}} (2-n)^{\frac{n}{n-2}} (n-1)^2 e^{f_2 n} z^3 \right. \\ & - 2g_0^{\frac{n}{n-2}} (2-n)^{\frac{n}{n-2}} (n-2)(n-1) e^{f_2 n} z \left. \right) h^{\frac{n}{n-2} + \frac{2}{n-2}} h' \\ & + \left( e^{f_2 n} \left( n^{\frac{n}{n-2}} (n^2 - 2n + 1) \right) h^{\frac{2}{n-2}} \right. \\ & + \left. \left( g_0^{\frac{n}{n-2}} (2-n)^{\frac{n}{n-2}} n - 2g_0^{\frac{n}{n-2}} (2-n)^{\frac{n}{n-2}} \right) h^{\frac{2n}{n-2}} \right) \\ & - \left( g_0^{\frac{n}{n-2}} (2-n)^{\frac{n}{n-2}} (n-1)^2 z^{\frac{2n^3}{n^2-3n+2}} \right) h^{\frac{n}{n-2} + \frac{2}{n-2}} (h')^n = 0. \end{aligned}$$

**Reduction 3.** For the generator  $\mathbf{v}_3^2$ , the similarity variable and similarity solution are

$$z = \frac{x}{t}, \quad u = -\frac{2}{c} \ln t + h(z),$$

where  $h(z)$  satisfies

$$(c^2 z^2 - c^2)h'' + (a z^2 c^2)(h')^2 + (4 z c a + 2 z c^2)h' + k e^{hc} c^2 + 4 a + 2 c = 0.$$

**Reduction 4.** For the generator  $\mathbf{v}_3^3$ , the invariant solution is

$$z = \frac{x}{t}, \quad u = \frac{1}{c} \left( t^{-\frac{2}{n-1}} h(z) - b \right),$$

where  $h(z)$  satisfies

$$\begin{aligned} & (n^2 z^2 - 2 n z^2 - n^2 + z^2 + 2 n) h'' - h'' h + (2 n^2 z - 2 z a) h' \\ & + (a n^2 - 2 a n + a) (h^n) + (2 n + 2) h = 0. \end{aligned}$$

**Reduction 5.** For the generator  $\mathbf{v}_3^4$ , the invariant solution is

$$z = \frac{x}{t}, \quad u = \frac{1}{c} \left( t^{-2} h(z) - k \right),$$

where  $h(z)$  satisfies

$$(z^2 - 1)h'' + \left( -\sqrt[3]{\frac{-h'z-2h}{c}} a z + 6z \right) h' + c h^2 + \left( -2 \sqrt[3]{\frac{-h'z-2h}{c}} a + 6 \right) h = 0.$$

#### 4. Traveling Wave Solutions

In this section we are studying Equation (6) in order to find traveling wave solutions of Equation (1). The other ordinary differential equations obtained are not considered because they are non-autonomous differential equations.

The procedure followed compares Equation (6) with a similar equation, studied before by Kudryashov, whose general solution appears in [31].

The second-order Equation (6) is expressed as

$$h'' = \frac{1}{1-\lambda^2} f(-\lambda h') + \frac{1}{1-\lambda^2} g(h). \quad (7)$$

Kudryashov obtained in [31] the general solution of a second-order ODE, given by

$$h'' = \frac{1}{\lambda} \left( \mu h' + \frac{1}{2} h^2 - \omega h - c_0 \right), \quad (8)$$

where  $c_0$  is an arbitrary constant and  $\lambda, \mu, \omega$  satisfies  $\omega = \frac{6\mu^2}{25\lambda}$ . The expression of its general solution is given in terms of the Weierstrass elliptic function, considering  $g_2 = 0$  and  $g_3 = c_1$ ,

$$h(z) = \omega_k + \frac{6\alpha^2}{25\beta} - \exp\left\{\frac{2z\alpha}{5\beta}\right\} \mathcal{P}\left(c_2 - \frac{5\beta}{\alpha\sqrt{12\beta}} \exp\left\{\frac{z\alpha}{5\beta}\right\}, 0, c_1\right),$$

with  $c_1, c_2$  arbitrary constants.

Equations (7) and (8) are equal if

$$\begin{aligned} f(-\lambda h') &= \frac{1-\lambda^2}{\lambda} \mu h', \\ g(h) &= \frac{1}{\lambda} \left( \frac{1}{2} h^2 - \omega h - c_0 \right). \end{aligned}$$

Consequently, the solutions of Equations (7) and (8) are equivalent for the previous expressions of  $f(-\lambda h')$  and  $g(h)$ .

Lastly, the following analytical solution of the original PDE (1) is given by undoing the change of variables (5):

$$u(x, t) = \omega + \frac{6\alpha^2}{25\beta} - \exp\left\{\frac{2(x-\lambda t)\alpha}{5\beta}\right\} \mathcal{P}\left(c_2 - \frac{5\beta}{\alpha\sqrt{12\beta}} \exp\left\{\frac{(x-\lambda t)\alpha}{5\beta}\right\}, 0, c_1\right). \quad (9)$$

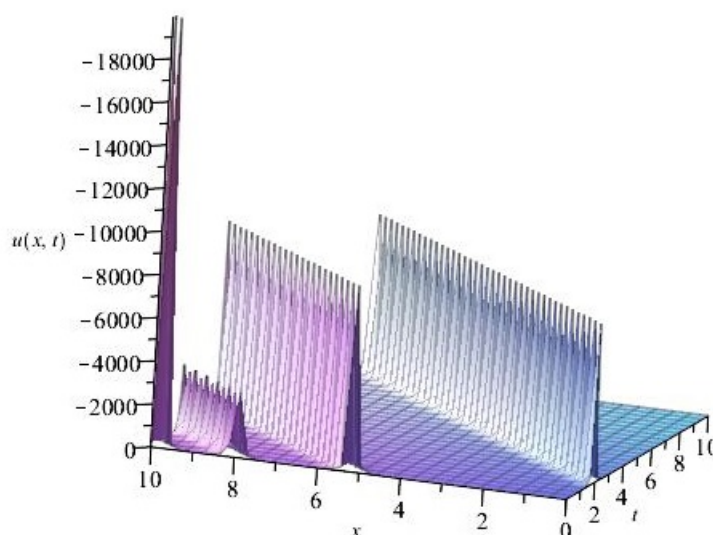


Figure 1. Solution (9), for  $\lambda = \alpha = \beta = c_1 = c_2 = 1$ .

Solution (9) is a soliton-type traveling wave solution (see Figure 1).

In addition, by the same procedure, the authors of [32] obtained the general solution of a second-order ODE of the form

$$h'' = \frac{2b(\beta q - p) - \alpha}{b(\beta^2 + 1)} h' + \frac{m_1 n_1}{b(\beta^2 + 1)} h^2 - \frac{b(p^2 + q^2) + a}{b(\beta^2 + 1)} h.$$

In the same way, we can derive the general solution for this equation, for

$$\begin{aligned} f(-\lambda h') &= \frac{2b(\beta q - p) - \alpha}{b(\beta^2 + 1)} (1 - \lambda^2) h', \\ g(h) &= \frac{m_1 n_1}{b(\beta^2 + 1)} (1 - \lambda^2) h^2 - \frac{b(p^2 + q^2) + a}{b(\beta^2 + 1)} (1 - \lambda^2) h. \end{aligned}$$

Furthermore, we can find another solution by using the Jacobi elliptic function method. Let us assume that Equation (6) has a solution of the form

$$h = \alpha H^\beta(z),$$

where  $\alpha$  and  $\beta$  are parameters to be determined. Here,  $H(z)$  is a solution of the Jacobi equation

$$(H')^2 = r + p H^2 + q H^4, \quad (10)$$

with  $r$ ,  $p$  and  $q$  constants.

Here  $H$  has the expression of an exponential or polynomial function. If  $H$  is a solution of Equation (10), then we can distinguish three cases: (i)  $H$  is the Jacobi elliptic sine function  $sn(z, m)$ ; (ii)  $H$  is the Jacobi elliptic cosine function,  $cn(z, m)$ ; (iii)  $H$  is the Jacobi elliptic function of the third kind  $dn(z, m)$ . However, we focus on the first case.

If  $H(z) = sn(z, m)$ ,

$$h(z) = p sn^q(z|m) \quad (11)$$

is a solution of Equation (6). Substituting Equation (11) into Equation (6), we obtain the expressions of  $f(-\lambda h')$ ,  $g(h)$ , and the parameters that make Equation (11) a solution of Equation (6).

This procedure was applied by Bruzón and Gandarias [33] to a similar equation, obtaining an exact solution. In the same way, an exact solution of Equation (1) is

$$u(x, t) = p sn^q(x - \lambda t|m). \quad (12)$$

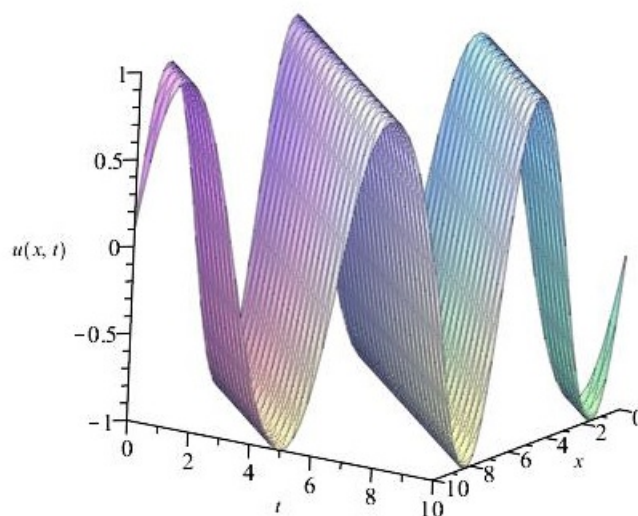


Figure 2. Solution (12) for  $\lambda = p = q = 1$  and  $m = 0.5$ .



Solution (12) is shown in Figure 2. Specifically, the solution is a stable non-linear non-harmonic oscillatory periodic wave.

## 5. Conservation Laws

The notion of a conservation law is a mathematical formulation of the familiar physical laws of conservation of energy, conservation of momentum and so on. This concept plays an important role in the analysis of basic properties of the solutions. For example, the invariance of a variational principle under a group of time translations implies the conservation of energy for the solutions of the associated Euler-Lagrange equations, and the invariance under a group of spatial translations implies conservation of momentum [19].

Anco and Bluman presented a direct conservation law method for PDEs expressed in normal form. A PDE is in normal form if it can be expressed in a solved form for some leading derivative of  $u$ , such that all the other terms in the equation contain neither the leading derivative nor its differential consequences [26].

A local conservation law of the non-linear viscoelastic wave Equation (1) satisfies the space-time divergence expression

$$D_t T + D_x X = (u_{tt} - u_{xx} + f(u_t) - g(u))Q, \quad (13)$$

named the characteristic equation, where  $T$  is the conserved density and  $X$  the conserved flux. The vector  $(T, X)$  is called the conserved current.

The general expression of a low-order multiplier  $Q$ , written in terms of  $u$  and derivatives of  $u$ , depends on those variables that, by derivatives, can lead to a leading derivative of Equation (1). For example,  $u_{tt}$  can be derived by the derivative of  $u_t$  with respect to  $t$ , and  $u_{xx}$  by the derivative of  $u_x$  with respect to  $x$ .

Therefore, it is defined

$$Q(t, x, u, u_t, u_x)$$

as the general expression for a low-order multiplier for the non-linear viscoelastic wave Equation (1).

However, all low-order multipliers are found by solving the determining equation

$$E_u((u_{tt} - u_{xx} + f(u_t) - g(u))Q) = 0, \quad (14)$$

where  $E_u$  is the Euler operator with respect to  $u$  [19], defined by

$$E_u = \partial_u - D_x \partial_{u_x} - D_t \partial_{u_t} + D_x D_t \partial_{u_{xt}} + D_x^2 \partial_{u_{xx}} + \dots$$

Hence, splitting the determining Equation (14) with respect to  $u_{xx}, u_{tt}, u_{tx}$ , we obtain an overdetermined linear system for  $Q, f(u_t), g(u)$ .

Thus, a complete classification of multipliers is found by solving the system with the same algorithmic method used for the determining equation for infinitesimal symmetries. The classification of multipliers is shown in Theorem 2. Then, for each multiplier we determine the corresponding conserved density  $T$  and flux  $X$ , by integrating directly the characteristic Equation (13). For this classification we apply the same Maple commands used for the Lie symmetries classification, “rfsimp”, “dsolve”, and “pdsolve”. Theorem 3 shows the results obtained.

**Theorem 2.** *All the multipliers admitted by the non-linear viscoelastic wave Equation (1), with  $f(u_t) \neq 0$ , are given below.*

1. For  $f(u_t) = f_0$  and  $g(u)$  arbitrary function, the multipliers are

$$Q_1 = u_t, \quad Q_2 = u_x.$$



2. For  $f(u_t) = f_0$  and  $g(u) = g_1 e^{g_0 u} - f_0$ , besides  $Q_1$  and  $Q_2$ , the multiplier is

$$Q_3 = tu_t + xu_x + \frac{2}{g_0}.$$

3. For  $f(u_t) = f_0 u_t + f_1$  and  $g(u)$  an arbitrary function, the multiplier is

$$Q_4 = u_x e^{f_0 t}.$$

4. For  $f(u_t) = -g_0 - \frac{1}{u_t f_0 + f_1}$  and  $g(u) = g_0$ , the multiplier is

$$Q_5 = f_0 u_t u_x + f_1 u_x.$$

5. For  $f(u_t) = -\frac{4f_0}{u_t + f_1} + f_2$  and  $g(u) = \frac{4f_0}{f_1} - f_2$ , the multiplier is

$$Q_6 = u_t + f_1.$$

**Theorem 3.** All the non-trivial low-order conservation laws admitted by the non-linear viscoelastic wave Equation (1), with  $f(u_t) \neq 0$ , are given below.

1. For  $f(u_t) = f_0$ ,  $g(u)$  arbitrary function and  $Q_1 = u_t$ , the conservation law is

$$\begin{aligned} T_1 &= \frac{1}{2} u_x^2 + \frac{1}{2} u_t^2 + \int g(u) + f_0 du, \\ X_1 &= -u_t u_x. \end{aligned} \quad (15)$$

2. For  $f(u_t) = f_0$ ,  $g(u)$  arbitrary function and  $Q_2 = u_x$ , the conservation law is

$$\begin{aligned} T_2 &= u_t u_x, \\ X_2 &= -\frac{1}{2} u_x^2 - \frac{1}{2} u_t^2 + \int g(u) + f_0 du. \end{aligned} \quad (16)$$

3. For  $f(u_t) = f_0$ ,  $g(u) = g_1 e^{g_0 u} - f_0$  and  $Q_3 = tu_t + xu_x + \frac{2}{g_0}$ , the conservation law is

$$\begin{aligned} T_3 &= \frac{1}{2g_0} 2te^{u g_0} g_1 + (tu_t^2 + tu_x^2 + 2u_x u_t) g_0 + 4u_t, \\ X_3 &= \frac{1}{2g_0} 2xe^{u g_0} g_1 + (-2tu_t u_x - xu_t^2 - u_x^2 x) g_0 - 4u_x. \end{aligned}$$

4. For  $f(u_t) = f_0 u_t + f_1$ ,  $g(u)$  an arbitrary function and  $Q_4 = u_x e^{f_0 t}$ , the conservation law is

$$\begin{aligned} T_4 &= u_x e^{f_0 t} u_t, \\ X_4 &= \int e^{f_0 t} (g(u) + f_1) du + \frac{1}{2} (-u_t^2 - u_x^2) e^{f_0 t}. \end{aligned}$$

5. For  $f(u_t) = -g_0 - \frac{1}{u_t f_0 + f_1}$ ,  $g(u) = g_0$  and  $Q_5 = f_0 u_t u_x + f_1 u_x$ , the conservation law is

$$\begin{aligned} T_5 &= \frac{1}{6} f_0 u_x^3 + \frac{1}{2} f_0 u_t^2 u_x + f_1 u_x u_t, \\ X_5 &= -\frac{1}{2} f_0 u_t u_x^2 - \frac{1}{2} u_x^2 f_1 - \frac{1}{6} f_0 u_t^3 - u - \frac{1}{2} u_t^2 f_1. \end{aligned}$$

6. For  $f(u_t) = -\frac{4f_0}{u_t + f_1} + f_2$ ,  $g(u) = \frac{4f_0}{f_1} - f_2$  and  $Q_6 = u_t + f_1$ , the conservation law is

$$\begin{aligned} T_6 &= \frac{1}{2} u_x^2 + \frac{1}{2} u_t^2 + f_1 u_t + 4 \frac{f_0 u}{f_1}, \\ X_6 &= (-u_t - f_1) u_x. \end{aligned}$$

Next, we study the meaning of some of these conservation laws. Every conservation law yields a corresponding conserved integral

$$\mathcal{C}[u] = \int_{\Omega} T \, dx,$$

where  $\Omega$  is the domain of solutions  $u(x, t)$ .

For Equation (1), with  $f(u_t)$  constant and  $g(u)$  non-linear function, conservation law (15) yields conservation of an energy quantity

$$\mathcal{E}[u] = \int_{\Omega} \frac{1}{2} u_x^2 + \frac{1}{2} u_t^2 + \int g(u) + f_0 \, du \, dx,$$

which represents the total energy for solution  $u(x, t)$ .

Conservation law (16) yields the conserved quantity

$$\mathcal{M}[u] = \int_{\Omega} u_t u_x \, dx,$$

which is a momentum quantity.

## 6. Conclusions

In this paper, we have obtained a complete Lie group classification for the viscoelastic wave Equation (1) in the presence of damping and source terms, for different expressions of the functions  $f$  and  $g$ . Then, we have constructed the corresponding reduced equations. These reductions make easier the resolution of the viscoelastic wave Equation (1) in order to obtain solutions of physical interest, such as solitons. Moreover, we have obtained these traveling wave solutions from the reduced equations by the comparison between Equation (1) and comparable equations studied before by other authors. Furthermore, Lie point symmetries are not the only ones that can be studied. Another symmetries such as contact or potential symmetries can be studied in the future. Finally, we have derived the non-trivial low-order conservation laws by using the multiplier method.

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